The Liar Paradox: Considering Fuzzy Logic and Trivalent Truth Conditions

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Introduction

- The term *self-reference* is used to denote any set of circumstances in which someone or something refers to itself.

- In philosophy, self-reference is primarily studied in the context of language.

- A *self-referential sentence*, then, is a sentence which refers to itself, and perhaps the most notable instance of this is the *liar sentence*.

Introduction (Cont.)

- Consider a sentence named (A), which says of itself (that is, says of (A)) that it is false:

  (A) Sentence (A) is false.

  (A) is self-referential because it refers to itself in the sentence. Additionally, it is *paradoxical* in the sense that it is self-contradictory. This self-contradictory nature of liar sentences is the primary motivator for what will follow in this presentation.

Introduction (Cont.)

- I will propose two distinct solutions to the *liar paradox*, one that involves fuzzy logic and another that involves supplying a third truth value to bivalent logic.

- I will also present *revenge paradoxes* for these solutions. Revenge paradoxes are arguments that arise in response to some proposed solution to a paradox that prove that solution to be insufficient.

- Overall, I will show that both solutions suffer from revenge paradoxes and that thus neither solution is satisfactory when dealing with the liar paradox.

Principle of Bivalence

- In "Function and Concept", Friedrich Ludwig Gottlob Frege offers his notion of a *function* (and, by extension, a *concept*) and demonstrates how he derives truth values of two types: true and false (Frege 137, 139).

- It is important to identify what suffices as truth values because we use them to assert propositions about given contexts or, more generally, the world.

Principle of Bivalence - Functions

- Frege defines a function as follows: "A function of x [is] taken to be a mathematical expression containing x, a formula containing the letter x. Thus…the expression $2x^3 + x$ would be a function of x, and $2[\times]2^3 + 2$ would be a function of 2." (Frege 131)

- Frege takes issue with the view that expressions such as "$1 + 3$" and "$2 + 2$" are equal but not the same because he believes that expressions, although able to vary in terms of form, retain some intimate feature about them known as content.

- Frege called this content *Bedeutung*, and this roughly translates to "reference".

Principle of Bivalence - Functions (Cont.)

- For example, Frege would state that, because both "1 + 3" and "2 + 2" are equal to four, both expressions retained the same *Bedeutung*.

- In the same vein, Frege supposes that expressions such as "2", "1 + 1", "3 — 1", and "6[/]3" all have the same *Bedeutung*. After all, there is no difference between the value, that being two, to which they all evaluate, but instead the only difference lies in their respective forms (Frege 132).

- Frege asserted that the fields of possible arguments and values for functions could be extended by adding signs such as (+), (—), etc., so that he could construct functional expressions, and also adding signs such as (=), (>), (<), etc., so that he could evaluate functions which are more comparative. All this leads us to Frege's notion of a concept.

Principle of Bivalence - Concepts

- Frege defined a concept as follows: “a concept is a function whose value is always a truth-value.” (Frege 139) But where does he recover what he considers “truth values” in the first place?

- Frege has us consider the function $x^2 = 1$, where x takes the place of some numeric argument, and evaluates it for different arguments:

  $$(-1)^2 = 1,\ 0^2 = 1,\ 1^2 = 1,\ 2^2 = 1.$$

  The first and third equations are true, whereas the second and fourth ones are false.

Principle of Bivalence - Concepts (Cont.)

- From this Frege shows that the value of a function can be a truth value and, if so, that such truth values come in two varieties: true and false. *True functions* are functions whose resultant values are consistent with what is asserted by their argument values, whereas *false functions* are functions whose resultant values fail to be consistent in this manner.

- Thus, equations such as “$2^2 = 4$”, for example, are true because two raised to the second power does, in fact, equate to four, whereas equations such as “$2^2 = 1$” are false because two raised to the second power does not equate to one. And as I stated before, Frege extended his terminology to more than just equations, and the aforementioned truth values also appear applicable in these cases.

Liar Paradox

- We now know that the *principle of bivalence* asserts that every sentence expressing a proposition has exactly one truth value, either true or false. But recall sentence (A):

  (A) Sentence (A) is false.

  The difficulty in applying this principle with regard to (A) arises from the fact that a contradiction emerges under both the supposition that (A) is true and the supposition that (A) is false.

Liar Paradox (Cont.)

- **Case 1:** Suppose that (A) is true. If (A) is true, then what (A) asserts must be true, and (A) asserts that (A) is false. If (A) is determined to be false, then it certainly cannot be true. So by supposing that (A) is true, we arrive at the conclusion that (A) is false, a contradiction.

  The converse can also be established.

- **Case 2:** Suppose that (A) is false. If (A) is false, then what (A) asserts must be false, and (A) asserts that (A) is false. In a case such as this, where the assertion that (A) is false is, in fact, false, then what (A) asserts of itself must actually be true. So by supposing that (A) is false, we arrive at the conclusion that (A) is true, again a contradiction.

Clearly, when we suppose that (A) is either true or false, we find that it ends up being both true and false. This is a paradox because propositions may be either true or false, but not both, under our current understanding. So how can we resolve this?

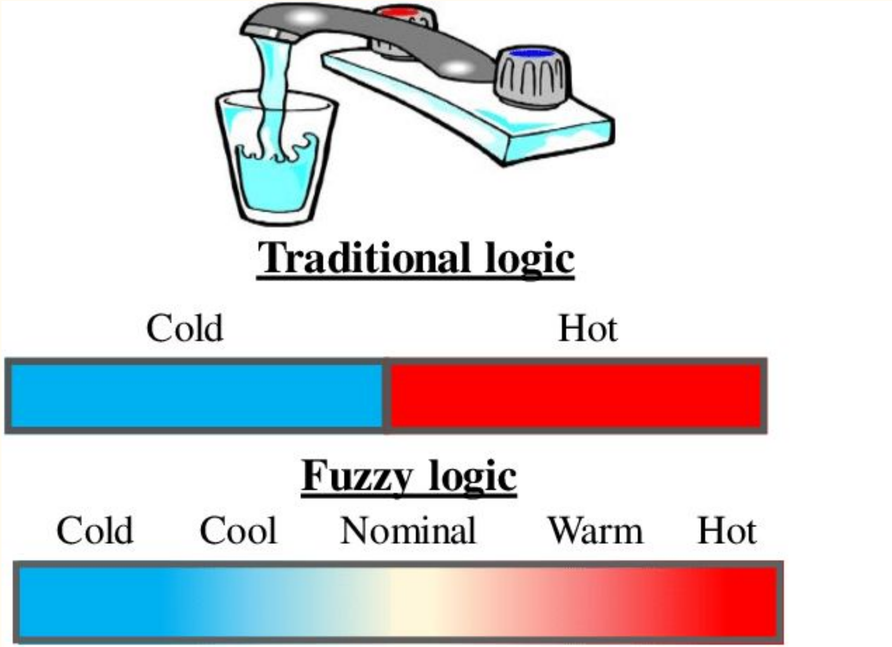
Fuzzy Logic

Fuzzy Logic

- Fuzzy logic is a form of many-valued logic in which the truth value of propositions may be defined by any real number between 0 and 1 (inclusive). It is employed to handle cases of partial truth, where the truth value may range between being entirely true and entirely false.

- For example, consider the temperature of tap water.

Fuzzy Logic (Cont.)



Fuzzy Logic (Cont.)

- In Boolean logic, where the truth values of propositions may only be defined by the integer values 0 (representing false) or 1 (representing true), we would represent the temperature of tap water as either cold or hot.

- With fuzzy logic, we could instead represent the temperature by means of a gradient from cold to hot. That is, we could account for arbitrary ranges of the temperature that fall between being cold and hot with terms such as “cool”, “nominal”, “warm”, etc., instead of limiting ourselves only to the prior two terms.

Fuzzy Logic (Cont.)

- As can be inferred, fuzzy logic is meant to be used to model logical reasoning in situations where the veracity of propositions is vague or imprecise, such as when the constraints of a particular proposition suggest that there are more outcomes to consider than just that of being entirely true or entirely false.

- Fuzzy logic is then a part of a family of many-valued logics where truth values are interpreted as degrees of truth instead of in accordance with the principle of bivalence.

Fuzzy Logic - Definitions

- Fuzzy logic works with membership values in a way that mimics Boolean logic, and Lotfi Zadeh's 1965 proposal of fuzzy set theory suggests that we should be able to determine the *complement* of a fuzzy set by relating it to Boolean logic (Zadeh 340).

- Under Boolean logic, the complement can be expressed as follows: NOT(x). We will need to concern ourselves with a fuzzy replacement for this.

Fuzzy Logic - Definitions (Cont.)

- The most important thing to realize about fuzzy logic is that it is a superset of conventional Boolean logic. That is, if we set the fuzzy logic values at their extremes of 1 (entirely true) and 0 (entirely false), then we will find that the standard logical operations from Boolean logic hold.

- In the following slide I have constructed the standard truth table for the NOT operator under Boolean logic.

Fuzzy Logic - Definitions (Cont.)

Truth Table for NOT Operator Under Boolean Logic

A	**not A**
0	1
1	0

- Considering that, under fuzzy logic, the truth of any proposition is a matter of degree, we will need to reframe this truth table such that it accounts for the range of possible degrees of truth that extend from 0 to 1. We can do this by replacing the operation NOT A with the operation $1 - A$. We will find that the previous truth table goes unaffected by this replacement.

Fuzzy Logic - Definitions (Cont.)

Truth Table for NOT Operator Under Fuzzy Logic

A	**1 — A**
0	1
1	0

- Thus, $1 - A$ suffices as the complement for A under fuzzy logic.

Fuzzy Logic - Initial Solution

- To successfully evaluate the liar sentence under fuzzy logic, we will need to interpret it under a more formal construction than how I have presented it thus far. Let x denote the degree of truth of sentence (A). If we want to determine x, then we must construct (A) in such a way that the choice of its possible truth values are restricted. I have shown that, so far, the only possible truth values that a sentence may evaluate to are true and false, so these will comprise the scope of truth values to which (A) will adhere.

Fuzzy Logic - Initial Solution (Cont.)

- Frege's conception of truth has it that sentences are referring terms. That is, they are names that name what is true and what is false. So one might suggest, if we had a Fregean version of fuzzy logic, that we might say that sentences are names of degrees of truth between 0 and 1. For example, it might make sense to say something like "$(1 + 3 = 4) = 1$" because "$1 + 3 = 4$" is just a name for the degree of truth 1, and so is "1".

Fuzzy Logic - Initial Solution (Cont.)

- On the view that sentences are names of degrees of truth, it makes sense to say, for example, "Sentence (A) is false = 0.5". After all, both sides of the equation contain a name of a degree of truth. However, it still will not make sense to say "(A) = 0.5". Under Frege's conception of truth, the left-hand side of the equation is a name of a *sentence*, not the name of a *degree of truth*. In other words, it is a name of a *name of a degree of truth*. If we want to use names of sentences to say something about degrees of truth, then we will need some device that allows us to take the names of sentences as arguments and map them onto degrees of truth.

Fuzzy Logic - Initial Solution (Cont.)

- Suppose that we have a degree function deg(x) that takes names of arguments and maps them onto degrees of truth. Because we want to evaluate (A) using this function, we end up with the degree function deg("Sentence (A) is false.") or, more simply, deg("False(A)"). Under Boolean logic, supposing that some proposition is false is equivalent to invoking the NOT operator onto it.

Fuzzy Logic - Initial Solution (Cont.)

- Thus, we end up with the following equation:

  $$\text{deg(A)} = \text{NOT(deg(A))}$$

  We can generalize the NOT operator to its equivalent under fuzzy logic:

  $$\text{deg(A)} = 1 - \text{deg(A)}$$

  We can rearrange this equation so that both deg(A)s are on the same side of the equation:

  $$2(\text{deg(A)}) = 1$$

  We can divide both sides of the equation by 2 to solve for deg(A):

  $$\text{deg(A)} = 0.5$$

So there appears to be a non-paradoxical solution to the liar paradox under fuzzy logic. Instead of treating the liar sentence as both entirely true and entirely false, we can treat it as precisely half-true and half-false.

Fuzzy Logic - Revenge Paradox

- As we all must recognize by now, fuzzy logic allows us to deal with many more truth values than Frege had ever anticipated existing. However, no issue should arise when we combine this framework with Frege's principle that substituting terms for coreferential (that is, having the same referent) ones should never change the truth value of a sentence.

- And yes, I realize that I did not explain how Frege arrived at this principle. Please take my word for its existence as it should intuitively make sense.

- Let us, then, consider two cases under the supposition that the fuzzy liar is true to some degree that is less than 1: one where the truth value of the fuzzy liar is (1) some degree that is less than 1 and another where the truth value is (2) equal to 1. It will be clear why the two cases result in very different problems.

Fuzzy Logic - Revenge Paradox (Cont.)

- In the first case, the fuzzy liar is true to some degree that is less than 1. More formally, we can express this as follows:

  (FL) $\deg(FL) < 1$.

  Suppose that the degree of truth of (FL) is equal to 0.5. Under this supposition, “deg(FL)” and “0.5” are coreferential terms. If Frege’s principle of substitution holds true, then we should be able to substitute terms for coreferential ones in (FL) without changing the truth value of (FL). That is, the following sentence should retain the same truth value as (FL):

  (FL*) $0.5 < 1$.

Fuzzy Logic - Revenge Paradox (Cont.)

- This sentence states that 0.5 is less than 1. It should be obvious that this is entirely true, and, under fuzzy logic, the truth value of something that is entirely true is 1. However, since we originally supposed that the degree of truth of (FL) was equal to 0.5, then this would suggest that this very same assertion is instead only half-true. Thus, a contradiction emerges where we are somehow equating 0.5 with 1. To put it more clearly, notice that, based on our supposition, all of the following will have to be coreferential:

  “deg(FL)”, “0.5”, “deg(FL*)”, “1”

  So by assigning the degree of truth 0.5 to (FL), we commit to the mathematical absurdity that 0.5 is equal to 1. And a similar argument will hold for any degree of truth of (FL) that is less than 1.

Fuzzy Logic - Revenge Paradox (Cont.)

- In the second case, the fuzzy liar is, again, true to some degree that is less than 1:

  (FL) $\deg(FL) < 1$.

  However, suppose that the degree of truth of (FL) is equal to 1 instead. Under this supposition, “deg(FL)” and “1” are coreferential terms. Again, if Frege’s principle of substitution holds true, then we should be able to substitute terms for coreferential ones in (FL) without changing the truth value of (FL). That is, the following sentence should retain the same truth value as (FL):

  (FL*) $1 < 1$.

Fuzzy Logic - Revenge Paradox (Cont.)

- Under fuzzy logic, something that has a truth value of 1 is entirely true. However, this sentence states that 1 is less than itself, and this cannot be true under any set of circumstances. In this case, then, this sentence must be entirely false. Thus, the degree of truth of (FL*) ought to be 0; not 1. So by assigning the degree of truth 1 to (FL), we are substituting a term for an allegedly coreferential one and thus altering the truth value of the sentence (unless $0 = 1$ or $1 < 1$, both of which are false).

- We are left wondering if there is any way that we can frame this issue such that it does not lead to mathematical absurdity.

Under the original liar paradox, fuzzy logic allows us to assert that the truth value of the liar sentence is 0.5. This would be an acceptable solution, since it resolves the contradiction of the liar sentence being both entirely true and entirely false by instead showing that it is half-true and half-false, had a revenge paradox not emerged from it. The revenge paradox I presented permits us to assert logically impossible conclusions, such as recognizably different values somehow equating with each other or the exact same value somehow equating with a value that is less than itself, so it does not seem as though we can actually establish a valid solution to the liar paradox under this approach.

Trivalent Truth Conditions

Trivalent Truth Conditions

- Another way that we might attempt to resolve the liar paradox involves introducing a third truth value, one which accounts for sentences that might best be described as indeterminate, a state of truth in which the veracity of some proposition cannot be precisely determined. Of course, justification is necessary for something like this.

- In the chapter “Three-valued logic: beginnings” of *Vagueness*, Timothy Williamson provides a reasonable justification for trivalent logic offered by Jan Łukasiewicz, who was concerned with free will: “fatalism can be avoided only if some statements about the future, such as ‘There will be a sea-fight tomorrow’, are not yet true or false.” (Williamson 102)

- This rationale should be intuitively pleasing to us. After all, how would we possibly be able to determine the truth value of something that has not yet occurred?

Trivalent Truth Conditions - Definitions

- In “Making Sense of (in)Determinate Truth: The Semantics of Free Variables”, John Cantwell provides definitions which allow us to understand indeterminateness more formally:

  (1) A_1, ..., A_n *semantically entail* B iff B is true relative to every assignment g where all the A_i are true.

  (2) In a context where all has been assumed is A_1, ..., A_n, an assignment g is *admissible* if and only if each A_i is true relative to g.

  (3) A sentence A is *determinately true (false)* in a context iff A is true (false) in every admissible assignment in that context.

  (4) A sentence A has *indeterminate truth value* in a context iff A is neither determinately true nor determinately false in that context. (Cantwell 2723–2724)

Trivalent Truth Conditions - Definitions (Cont.)

- We can establish two conclusions from these definitions. First, while assignment succeeds when some general solution includes *at least one* possible solution to some particular function, entailment only succeeds when some general solution includes *all* possible solutions to it. Second, because some sentence x is determinately true or determinately false only if it accounts for *all* possible solutions given a particular context, the sentence is indeterminate if it fails to account for *at least one* possible solution.

Trivalent Truth Conditions - Initial Solution

- Recall sentence (A):

  (A) Sentence (A) is false.

  To show that this sentence would be more appropriately described as indeterminate, I will need to demonstrate that the truth value of this proposition never ends up being only true or only false.

For “Sentence (A) is false” to be determinately true or determinately false, then it must be either true or false in every admissible assignment in that context, that is to say it must be only true or only false in all possible cases. We already know that if we suppose that (A) is true, then (A) will end up being false, and that if we suppose that (A) is false, then (A) will end up being true. The fact that the sentence always ends up being true or false does *not* make it both determinately true and determinately false. Again, for something to be determinately true or determinately false, only *one* of the truth values may hold for all cases. “Sentence (A) is false” is not true or false in all possible cases, so it is not determinately true or determinately false. And if “Sentence (A) is false” is not determinately true or determinately false, then it must retain an indeterminate truth value.

Trivalent Truth Conditions - Revenge Paradox

- Consider a variation of sentence (A) where the sentence is *not true* instead of false:

  (A*) Sentence (A*) is not true.

  One might initially wonder what exactly differentiates sentence (A) from sentence (A*). The answer is that sentence (A*)'s scope of truth values is more broad than sentence (A)'s because it not only captures that the sentence under investigation is false, but that the sentence is *either* false or indeterminate.

Trivalent Truth Conditions - Revenge Paradox (Cont.)

- **Case 1:** Suppose that (A*) is true. If (A*) is true, then what (A*) asserts must be true, and (A*) asserts that (A*) is not true, meaning that (A*) is either false or indeterminate. If (A*) is determined to be either false or indeterminate, then it certainly cannot be true. So by supposing that (A*) is true, we arrive at the conclusion that (A*) is either false or indeterminate, a contradiction.

- **Case 2:** Now, suppose that (A*) is false. If (A*) is false, then what (A*) asserts must be false, and (A*) asserts that (A*) is not true, again meaning that (A*) is either false or indeterminate. In a case such as this, where the assertion that (A*) is not true is, in fact, not true, then what (A*) asserts of itself must actually be true. So by supposing that (A*) is false, we arrive at the conclusion that (A*) is true, again a contradiction. And this line of reasoning holds true under the supposition that sentence (A*) is indeterminate as well.

Trivalent Truth Conditions - Revenge Paradox (Cont.)

- **Case 3:** Suppose that (A*) is indeterminate. If (A*) is indeterminate, then what (A*) asserts must be indeterminate, and (A*) asserts that (A*) is not true, again meaning that (A*) is either false or indeterminate. In a case such as this, where the assertion that (A*) is not true is, in fact, not true, then what (A*) asserts of itself must actually be true.

- So we are back in a position where all our suppositions lead to a contradiction. This suggests that introducing further truth values is unlikely to aid us in resolving the liar paradox.

Under the original liar paradox, trivalent logic allows us to assert that the truth value of the liar sentence is indeterminate. If this holds true, then the liar sentence cannot be said to be either determinately true or determinately false. So instead of us having to deal with the contradiction that arises from the liar sentence being both entirely true and entirely false, we could withhold judgment and not have to concern ourselves with the contradiction at all. However, the revenge paradox I presented has it that the liar sentence is more broad to the point such that its being indeterminate even leads to a contradiction, so, like with fuzzy logic, it does not seem as though we can actually establish a valid solution to the liar paradox under this approach either.

Conclusion

- In this presentation I have shown that the liar paradox seemingly prevails under all three systems of bivalent logic, fuzzy logic, and trivalent logic.

- Bivalent logic simply fails to have a resolution to the contradiction entailed by the liar paradox, whereas both fuzzy logic and trivalent logic succumb to revenge paradoxes which ultimately invalidate them as proper solutions. That is, none of the approaches seem to sufficiently resolve the liar paradox.

- Despite the fact that none of them prove to be successful in resolving the liar paradox, the approach involving fuzzy logic seems the *most* satisfactory because we at least end up deriving a truth value for the original liar paradox which makes intuitive sense (although, we still ultimately end up with mathematical absurdity).

Conclusion (Cont.)

- The lesson to be found here does not seem to be that liar sentences are neither true nor false, but instead that, in order to reflect as many of our intuitions as possible, paradoxical sentences must somehow be permitted without our theory of truth failing. And while it seems as though discarding bivalence provides us with one step toward doing this, it is still unclear what, exactly, is the right way to approach liar sentences.

Thank You!

- I would like to thank both Jason Decker and Allison Murphy for guiding me through this investigation from its inception to its completion.

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Questions?